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THE TWO-DIMENSIONAL LOADING PROBLEM OF AN ELASTO-PLASTIC PLANE WEAKENED BY A HOLE*

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A general approach to the solution of problems on finding the elasto-plastic boundaries of a plane under tension weakened by a hole, is considered. The propagation of the plastic domain in the plane is studied during the loading process, when part of the hole outline is in the elastic zone. A method is developed for carrying the matching conditions of the solutions from the elasto-plastic domain over to the hole contour, which would enable the determination of the elasto-plastic boundary to be reduced to the solution of a sequence of boundary value problems of elasticity theory. The possibility of applying the method developed to the problem of the uniaxial tension of an elasto-plastic plane weakened by a circular hole is analysed.

Many elasto-plastic problems in which the whole body contour is enclosed by the plastic domain /1, 2/ are solved by the method of perturbations. If the plastic flow starts from a certain point of the body contour, this method requires some modification /3/. The exact solution of the problem of the biaxial tension of an elasto-plastic plane weakened by a circular hole is obtained in /4/.

1. The plane state of strain of a loaded elasto-plastic plane weakened by a cylindrical hole with generators parallel to the x_3 axis is considered. Let L be the contour of the hole in the x_1x_2 plane.

The equilibrium equations that have the following form in the absence of mass forces:

$$\sigma_{\alpha\beta,\beta}=0 \tag{1.1}$$

hold in the elastic and plastic domains of the plane.

It is assumed that the hole boundary is stress-free, while forces P_1 and P_2 $(P_2 \geqslant P_1)$ are applied, respectively, to the plane at infinity parallel to the x_1 and x_2 axes of a Cartesian rectangular coordinate system

$$\sigma_{\alpha\beta}n_\alpha n_\beta|_L=\sigma_{\alpha\beta}n_\alpha t_\beta|_L=0, \quad \sigma_{11}^\infty=P_1, \sigma_{22}^\infty=P_2 \tag{1.2}$$

where n_α, t_α $(\alpha=1, 2)$ are unit vector components of the normal and tangent to the hole outline.

The solution of the problem under consideration can be found in the elastic domain by means of the Kolosov-Muskhelishvili formulas /5/

$$\sigma_{11}+\sigma_{22}=4 \operatorname{Re}\Phi(z), \ \sigma_{22}-\sigma_{11}+2i\sigma_{12}=2[\bar{z}\Phi'(z)+\Psi(z)] \tag{1.3}$$

where $\Phi(z)$ and $\Psi(z)$ are certaom analytic functions of the argument $z=x_1+ix_2$.

The stresses originating in the plastic domains of the plane satisfy the Tresca plasticity condition

$$s_{\alpha\beta}s_{\alpha\beta}=k^2; \quad s_{\alpha\beta}=\sigma_{\alpha\beta}-\sigma\delta_{\alpha\beta}, \quad \sigma=\sigma_{\alpha\alpha}/2 \tag{1.4}$$

If φ is the angle between the first principal stress and the x_1 axis, then taking account of (1.4) the relation between the stress components and the principal stresses in the plastic domain is /1/

$$\left.\begin{matrix}\sigma_{11}\\ \sigma_{22}\end{matrix}\right\}=\sigma\pm k\cos 2\varphi, \quad \sigma_{12}=k\sin 2\varphi, \quad k=\frac{1}{2}(\sigma_1-\sigma_2) \tag{1.5}$$

Satisfaction of the condition of continuity for the stress components

$$[\sigma_{\alpha\beta}]|_{L_s}=0 \tag{1.6}$$

is assumed on the boundary L_s of the elastic and plastic domains.

The relationships (1.1)-(1.6) completely define the state of stress of a plane with a hole for biaxial tension.

2. For certain values of the tensile forces $P_1=P_{10}=\text{const}$, $P_2<P_{20}$, let a plane

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weakened by a hole be in the elastic state, and for $P_2=P_{20}$ let at least one point A of the contour L exist at which condition (1.4) will be satisfied. The rectangular Cartesian coordinate system in Fig.1 is displayed so that the x_1 axis passes through this point perpendicular to the tangent drawn to L at the point A. For $P_2>P_{20}$ a plastic zone is formed around this point.

In parametric form the equation of the boundary L_s can be written as

$$x_\alpha(\theta)=x_\alpha^{(0)}(\theta)+r(\theta)\,n_\alpha \tag{2.1}$$

where θ is the slope of the unit normal drawn to the contour L to the x_1 axis, $x_\alpha^{(0)}(\theta)$ $(\alpha=1,2)$ are the coordinates of points of the contour L, and $r(\theta)$ is the magnitude of the segment along the normal to the boundary L with origin at L.

Taking account of (2.1) the conjugate condition for solutions (1.6) can be written in the form

$$\sum_{m=0}^{\infty}\frac{r^m}{m!}\frac{\partial^m\sigma_{\alpha\beta}^e}{\partial n^m}=\sum_{k=0}^{\infty}\frac{r^k}{k!}\frac{\partial^k\sigma_{\alpha\beta}^p}{\partial n^k}\quad\text{on}\quad L \tag{2.2}$$

where $\sigma_{\alpha\beta}^e$ and $\sigma_{\alpha\beta}^p$ are stress components in the elastic and plastic domains, respectively.

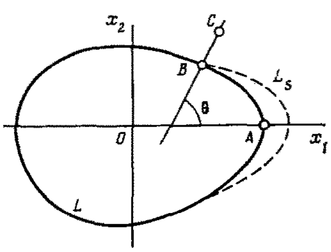

Fig.1

Therefore, the boundary conditions (1.2) and (2.2) in combination with (1.5) enable us to solve the equilibrium equations and to determine the stress components $\sigma_{\alpha\beta}^e$, $\sigma_{\alpha\beta}^p$ as well as the unknown function $r(\theta)$.

Let $P_1=P_{10}=0$ and the magnitude of the tensile force $P_2=P$ vary as a certain small parameter ε increases

$$P=P_0+\sum_{i=1}^{\infty}\varepsilon^iP_i,\quad P_0=P_{20} \tag{2.3}$$

Since the position of the boundary L_s depends on the quantity P, it is then natural to seek it in the form

$$x_\alpha=x_\alpha^{(0)}+n_\alpha\sum_{i=1}^{\infty}\varepsilon^ir_i \tag{2.4}$$

The stress components in the elastic domain are represented for $P>P_0$ in the form

$$\sigma_{\alpha\beta}^e=\sigma_{\alpha\beta}^{(0)}+\sum_{i=1}^{\infty}\varepsilon^i\sigma_{\alpha\beta}^{(i)} \tag{2.5}$$

where $\sigma_{\alpha\beta}^{(0)}$ are the stress components corresponding to the tensile force P_0.

Let $x_\alpha^{(0)}(\theta_*)$ denote the coordinates of the points of intersection of the boundary L_s for $P>P_0$ and the hole outline L. If the external loading process is sufficiently smooth and the transition of the elastic work of the material of the plane to the plastic work is smooth, then it can be assumed that the angle θ_* of enclosure of the contour L by the plastic zone will be a quantity of the order of ε.

The equilibrium Eqs.(1.1) are written in orthogonal curvilinear coordinates x and y in the form

$$\frac{\partial\sigma_x}{\partial x}+H\frac{\partial\tau_{xy}}{\partial y}+2\frac{\tau_{xy}}{R}=0,\quad \frac{\partial\tau_{xy}}{\partial x}+H\frac{\partial\sigma_y}{\partial y}+\frac{\sigma_y-\sigma_x}{R}=0 \tag{2.6}$$
$$\sigma_x=\sigma_{xx},\quad \sigma_y=\sigma_{yy},\quad \sigma_{xy}=\tau_{xy},\quad H=1+y/R$$

Here the arclength AB of the contour L (Fig.1) is taken as the coordinate x while the length of the normal BC is taken as the y coordinate.

The stress components (1.5) in the plastic zone are represented in the new coordinate system in the form

$$\left.\begin{matrix}\sigma_y\\ \sigma_x\end{matrix}\right\}=\sigma\pm k\cos2(\varphi-\theta),\quad \tau_{xy}=k\sin2(\varphi-\theta) \tag{2.7}$$

It will later be necessary to replace the angle φ by the angle $\varphi-\theta$, while preserving its previous notation, when going from the x_1x_2 coordinate system to the xy system.

Inserting the relationships (2.7) into (2.6), we obtain after reduction

$$\frac{\partial\sigma}{\partial x}-2k\left[\left(\frac{\partial\varphi}{\partial x}-\frac{1}{R}\right)\sin2\varphi-H\frac{\partial\varphi}{\partial y}\cos2\varphi\right]=0 \tag{2.8}$$
$$H\frac{\partial\sigma}{\partial y}+2k\left[\left(\frac{\partial\varphi}{\partial x}-\frac{1}{R}\right)\cos2\varphi+H\frac{\partial\varphi}{\partial y}\sin2\varphi\right]=0$$

In the notation used the equation of the elasto-plastic boundary (2.4) is written as:

$$x = x^{(0)}, \quad y = \sum_{i=1}^{\infty} \varepsilon^i r_i = \varepsilon \bar{r} \tag{2.9}$$

The equilibrium equations are valid everywhere in the elastic domain, including the boundary L_s, and therefore, by using the relationships (2.5) and (2.9), Eqs.(2.6) can be represented in the form

$$\begin{gathered}\left\{\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{m!}\left[\frac{\partial^{m+1}\sigma_x^{(k)}}{\partial x\partial y^m}+\left(1+\frac{\varepsilon\bar{r}}{R}\right)\frac{\partial^{m+1}\tau_{xy}^{(k)}}{\partial y^{m+1}}+\right.\right.\\ \left.\left.\frac{2}{R}\frac{\partial^m\tau_{xy}^{(k)}}{\partial y^m}\right]\varepsilon^{m+k}\bar{r}^m\right\}\Big|_L=0\\ \left\{\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{m!}\left[\frac{\partial^{m+1}\tau_{xy}^{(k)}}{\partial x\partial y^m}+\left(1+\frac{\varepsilon\bar{r}}{R}\right)\frac{\partial^{m+1}\sigma_y^{(k)}}{\partial y^{m+1}}+\right.\right.\\ \left.\left.\frac{1}{R}\left(\frac{\partial^m\sigma_y^{(k)}}{\partial y^m}-\frac{\partial^m\sigma_x^{(k)}}{\partial y^m}\right)\right]\varepsilon^{m+k}\bar{r}^m\right\}\Big|_L=0\end{gathered} \tag{2.10}$$

Eqs.(2.8) hold everywhere in the plastic domain including the boundary L_s.

The following equality can be considered the result of the assumption about the order of smallness of the magnitude of the angle θ_* of enclosure by the plastic domain:

$$x_*^{(0)} = o(\varepsilon) \tag{2.11}$$

Taking account of (2.7) and (2.9), we can write (2.8) to $O(\varepsilon)$ accuracy in the form

$$\left\{\frac{\partial\sigma}{\partial x} + 2k\frac{\partial\varphi}{\partial y}\right\}\Big|_L = 0, \quad \left\{\frac{\partial\sigma}{\partial y} + 2k\left(\frac{\partial\varphi}{\partial x} - \frac{1}{R}\right)\right\}\Big|_L = 0 \tag{2.12}$$

Along the contour L free from an external load the magnitude of σ is constant, from which by taking account of the plasticity condition we conclude

$$\sigma|_L = k \tag{2.13}$$

There results from the condition that the angle φ is constant along the load-free hole contour

$$\partial\varphi/\partial x\,|_L = 0 \tag{2.14}$$

According to (2.13) and (2.14), Eqs.(2.12) take the following form to $O(\varepsilon)$ accuracy:

$$\partial\varphi/\partial y\,|_L = 0, \quad \partial\sigma/\partial y\,|_L = 2k/R \tag{2.15}$$

Taking account of (2.9) and (2.15), the stress components along the boundary L_s can be written as follows to $O(\varepsilon^2)$ accuracy from the plastic domain side

$$\sigma_y = \varepsilon 2kr_1/R, \quad \sigma_x = 2k + \varepsilon 2kr_1/R, \quad \tau_{xy} = 0 \tag{2.16}$$

Taking (2.5), (2.9) and (2.16) into account, the matching condition of the solutions (1.6) in the xy coordinate system are rewritten to $O(\varepsilon^2)$ accuracy in the form of the equalities

$$\begin{gathered}\sigma_y^{(1)}|_L = -r_1\left(\frac{\partial\sigma_y^{(0)}}{\partial y}\Big|_L - \frac{2k}{R}\right), \quad (\sigma_x^{(0)}|_L - 2k) + \varepsilon G_x^{(1)}|_L = \\ -\varepsilon r_1\left(\frac{\partial\sigma_x^{(0)}}{\partial y}\Big|_L - \frac{2k}{R}\right), \quad \tau_{xy}^{(1)}|_L = -r_1\frac{\partial\tau_{xy}^{(0)}}{\partial y}\Big|_L\end{gathered} \tag{2.17}$$

We examine the term $F(x) = \sigma_x^{(0)}|_L - 2k$ on the left-hand side of the second equality in (2.17). By the assumption made above, a plastic state is realized at the point A of the contour L for $P = P_0$. At the same point $\sigma_x^{(0)}$ takes its maximum value on L, i.e.,

$$F(0) = (\sigma_x^{(0)} - 2k)|_{x=0} = 0, \quad \frac{\partial F}{\partial x}\Big|_{x=0} = \frac{\partial\sigma_x^{(0)}}{\partial x}\Big|_{x=0} = 0 \tag{2.18}$$

Taking these equalities into account, the Taylor series for the function $F(x)$ can be written in the neighbourhood of the point A in the following form:

$$F(x) = \sum_{m=2}^{\infty}\frac{1}{m!}\frac{\partial^m F}{\partial x^m}\Big|_{x=0} x^m = o(\varepsilon^2) \tag{2.19}$$

since x in relationship (2.19) lies on the arc $[0; x_*^{(0)}]$.

To determine the order of smallness of the other quantities in the boundary conditions

(2.17) it is necessary to consider (2.10) for the zeroth approximation $(\varepsilon = 0)$

$$\left\{\frac{\partial \sigma_x^{(0)}}{\partial x} + \frac{\partial \tau_{xy}^{(0)}}{\partial y}\right\}\Big|_L = 0, \quad \left\{\frac{\partial \tau_{xy}^{(0)}}{\partial x} + \frac{\partial \sigma_y^{(0)}}{\partial y}\right\}\Big|_L = \frac{2k}{R} \tag{2.20}$$

From the first relationship in (2.18) and the assumption (2.11) it follows that: $\partial\sigma_x^{(0)}/\partial x\,|_L = o(\varepsilon)$, from which according to the first equation in (2.20) $\partial\tau_{xy}^{(0)}/\partial y\,|_L = o(\varepsilon)$. Since the side surface of the cylindrical hole is stress-free, the equality $\partial\tau_{xy}^{(0)}/\partial x\,|_L = 0$ is valid, and taking it into account we obtain from the second equaion in (2.20) $\partial\sigma_y^{(0)}/\partial y\,|_L - 2k/R = o(\varepsilon)$.

Therefore, to an accuracy of $O(\varepsilon)$ the boundary conditions (2.17) can be written in the form

$$\sigma_y^{(1)}|_L = 0, \quad \sigma_x^{(1)}|_L = -r_1\left(\frac{\partial \sigma_x^{(0)}}{\partial y} - \frac{2k}{R}\right)\Big|_L, \quad \tau_{xy}^{(1)}|_L = 0 \tag{2.21}$$

The first and third conditions in (2.21) in combination with the condition at infinity determine the state of stress of the material of the plane in the elastic domain

$$\sigma_x^{(1)} = h\sigma_x^{(0)}, \quad \sigma_y^{(1)} = h\sigma_y^{(0)}, \quad \tau_{xy}^{(1)} = h\tau_{xy}^{(0)}, \quad h = \text{const}$$

Substitution of these solutions into the second boundary condition (2.21) yields

$$r_1 = 2kh\left(\frac{\partial \sigma_x^{(0)}}{\partial y}\Big|_L - \frac{2k}{R}\right)^{-1} = \text{const} + O(\varepsilon) \tag{2.22}$$

Comparing relationships (2.11) and (2.22) we obtain

$$r_1 = P_1 = h = 0 \tag{2.23}$$

Therefore, under the assumption (2.11) made relative to the order of smallness of $x_*^{(0)}$ the expansion of the desired solutions in series in the small parameter starts with the second power of ε.

Without loss of generality we can set

$$P = P_0(1 + \varepsilon^2) \tag{2.24}$$

thereby defining the small loading parameter which had been undefined in all the previous expansions.

Using relationships (2.23) and reasoning as before, the stress tensor components along L_s from the plastic domain side can be written with $O(\varepsilon^3)$ accuracy in the form

$$\sigma_y = \varepsilon^2 2kr_2/R, \quad \sigma_x = 2k + \varepsilon^2 2kr_2/R, \quad \tau_{xy} = 0$$

Taking these equalities into account, the matching condition of the solutions (1.6) are written to $O(\varepsilon^3)$ accuracy in the form

$$\sigma_y^{(2)}|_L = 0, \quad \left\{\frac{1}{\varepsilon^2}(\sigma_x^{(0)} - 2k) + \sigma_x^{(2)}\right\}\Big|_L = -r_2\left(\frac{\partial \sigma_x^{(0)}}{\partial y} - \frac{2k}{R}\right)\Big|_L, \tag{2.25}$$
$$\tau_{xy}^{(2)}|_L = 0$$

The first and third boundary conditions in (2.25) in combination with (2.24) determine the state of stress of the material of the plane in the elastic zone

$$\sigma_y^{(2)} = \sigma_y^{(0)}, \sigma_x^{(2)} = \sigma_x^{(0)}, \tau_{xy}^{(2)} = \tau_{xy}^{(0)}$$

The second condition in (2.25) enables us to find the function

$$r_2 = \left(\frac{\sigma_x^{(0)}|_L - 2k}{\varepsilon^2} + \sigma_x^{(2)}|_L\right)\left(\frac{2k}{R} - \frac{\partial \sigma_x^{(0)}}{\partial y}\Big|_L\right)^{-1}$$

Therefore, taking account of the assumptions made, the problem of the uniaxial tension of an elasto-plastic plane weakened by a cylindrical hole can be reduced to a set of elasticity theory boundary value problems. The boundary conditions for the sequence of elastic problems are determined by using the matching condition for the solutions on L_s. They can be written as follows in general form

$$\sigma_y^{(i+1)}|_L = f_1^{(i+1)}(r_i, \varepsilon, x) \tag{2.26}$$
$$\sigma_x^{(i+1)}|_L = f_2^{(i+1)}(r_{i+1}, \varepsilon, x)$$
$$\tau_{xy}^{(i+1)}|_L = f_3^{(i+1)}(r_i, \varepsilon, x), \quad i = 2, 3, \ldots$$

$$\sigma_y^{(2)} = \sigma_y^{(0)}, \quad \sigma_x^{(2)} = \sigma_x^{(0)}, \quad \tau_{xy}^{(2)} = \tau_{xy}^{(0)}$$

The first and third boundary conditions in (2.26) determine the state of stress of the material of the plane in the solution of the corresponding elastic problem; the desired function r_{i+1} is found by using the second boundary condition.

A constant, which is a result of the static determinancy of this problem, is imposed on the solution constructed for the problem under consideration: each plastic element of the plane should be connected to the hole contour by slip lines that lie entirely in the plastic zone.

We also note that the series expansions in the small loading parameter introduced above are not Taylor series expansions since the dependence of $x_*^{(0)}$ on ε is not taken into account, i.e., the expansions constructed are formal. The following reasoning justifies such an expansion: the solutions being obtained in the elastic and plastic domains satisfy the theories of elasticity and plasticity as well as the boundary conditions; only the conditions on the elasto-plastic boundary are satisfied approximately, consequently, the differences in the stresses on the boundary L_s were determined numerically for the solution constructed in the example examined below.

3. A problem of biaxial tension of a plane weakened by a circular hole of unit radius is considered. The solution of the elastic problem with boundary conditions (1.2) is sought for $P_1 = P_{10}$, $P_2 = P_{20}$ $(P_{20}/P_{10} \gg 1)$ by using (1.3) and will be /5/

$$\begin{aligned}
&\sigma_r^{(0)} = \frac{1}{2} A\left(1 - \frac{1}{r^2}\right) - \frac{1}{2} B\left(1 + 3\frac{1}{r^4} - 4\frac{1}{r^2}\right)\cos 2\theta \\
&\sigma_\theta^{(0)} = \frac{1}{2} A\left(1 + \frac{1}{r^2}\right) + \frac{1}{2} B\left(1 + 3\frac{1}{r^2}\right)\cos 2\theta \\
&\tau_{r\theta}^{(0)} = \frac{1}{2} B\left(1 + 2\frac{1}{r^2} - 3\frac{1}{r^4}\right)\sin 2\theta; \quad A = P_{20} + P_{10}, \\
&B = P_{20} - P_{10}
\end{aligned} \tag{3.1}$$

Let the conditions at infinity be given in the form

$$\sigma_{11}^\infty = P_{10}(1 + \alpha\varepsilon^2), \quad \sigma_{22}^\infty = P_{20}(1 + \varepsilon^2), \quad \alpha \in [0, 1] \tag{3.2}$$

The stresses (3.1) satisfy the plasticity condition (1.4) at two points of the contour L with the coordinates $(1, 0)$, $(1, \pi)$ for $P_2 = P_{20}$, $P_1 = P_{10}$ $(3P_{20} - P_{10} = 2k)$. For $P_2 > P_{20}$ (3.2) plastic domains are formed around these points, wherein the stress components are determined by the equalities /1/

$$\sigma_r^p = 2k \ln r, \quad \sigma_\theta^p = 2k + \sigma_r^p, \quad \tau_{r\theta}^p = 0 \tag{3.3}$$

Taking account of (2.23) in polar coordinates, we write Eq.(2.4) of the boundary L_s in the form

$$r_s = 1 + \varepsilon^2 r_2 + \varepsilon^3 r_3 + O(\varepsilon^4) \tag{3.4}$$

The angle θ_* of hole contour enclosure by the plastic domain is determined from the condition

$$r_s(\varepsilon, \theta_*) = 1, \quad \sin\theta_* = o(\varepsilon) \tag{3.5}$$

According to (3.3) and (3.4) the stress components $\sigma_{\alpha\beta}^p$ along the elasto-plastic boundary will be the following

$$\sigma_r^p|_{L_s} = 2k[\varepsilon^2 r_2 + \varepsilon^3 r_3 + O(\varepsilon^4)], \quad \sigma_\theta^p|_{L_s} = 2k[1 + \varepsilon^2 r_2 + \varepsilon^3 r_3 + O(\varepsilon^4)], \quad \tau_{r\theta}^p|_{L_s} = 0 \tag{3.6}$$

Taking (2.5), (3.4) and (3.6) into account, the matching conditions for solutions (1.6) can be written in the form

$$\begin{aligned}
&\left\{\sigma_r^{(0)} + \varepsilon^2\left(\sigma_r^{(2)} + \frac{\partial\sigma_r^{(0)}}{\partial r} r_2\right) + \varepsilon^3\left(\sigma_r^{(3)} + \frac{\partial\sigma_r^{(0)}}{\partial r} r_3\right) + O(\varepsilon^4)\right\}\Big|_{L_*} = \\
&\quad 2k[\varepsilon^2 r_2 + \varepsilon^3 r_3 + O(\varepsilon^4)] \\
&\left\{\sigma_\theta^{(0)} + \varepsilon^2\left(\sigma_\theta^{(2)} + \frac{\partial\sigma_\theta^{(0)}}{\partial r} r_2\right) + \varepsilon^3\left(\sigma_\theta^{(3)} + \frac{\partial\sigma_\theta^{(0)}}{\partial r} r_3\right) + O(\varepsilon^4)\right\}\Big|_{L_*} = \\
&\quad 2k[1 + \varepsilon^2 r_2 + \varepsilon^3 r_3 + O(\varepsilon^4)] \\
&\left\{\tau_{r\theta}^{(0)} + \varepsilon^2\left(\tau_{r\theta}^{(2)} + \frac{\partial\tau_{r\theta}^{(0)}}{\partial r} r_2\right) + \varepsilon^3\left(\tau_{r\theta}^{(3)} + \frac{\partial\tau_{r\theta}^{(0)}}{\partial r} r_3\right) + O(\varepsilon^4)\right\}\Big|_{L_*} = 0
\end{aligned} \tag{3.7}$$

where L_* is the part of the hole contour in the plastic zone.

The boundary conditions to find the second approximation of the solution of (1.1) in the elastic domain are determined by (3.1), (3.2), (3.5) and (3.7) and will be

$$\sigma_r^{(2)}|_{r=1}=0,\quad \tau_{r\theta}^{(2)}|_{r=1}=0,\quad \sigma_{11}^{\infty}=\alpha P_{10},\quad \sigma_{22}^{\infty}=P_{20} \tag{3.8}$$
$$\sigma_\theta^{(2)}|_{r=1}-4(P_{20}-P_{10})\varepsilon^{-2}\sin^2\theta=2r_2(5P_{20}-3P_{10})$$

The solution of the problem formulated with the first four boundary conditions in (3.8) has the form (3.1) for $A=P_{20}+\alpha P_{10}$, $B=P_{20}-\alpha P_{10}$. The unknown function r_2 is determined from the fifth condition in (3.8)

$$r_2=\frac{s_0}{2s_2}-2\frac{s_1}{s_2}\frac{\sin^2\theta}{\varepsilon^2},\quad s_0=3P_{20}-\alpha P_{10},\quad s_1=P_{20}-P_{10}, \tag{3.9}$$
$$s_2=5P_{20}-3P_{10}$$

The first approximation $\theta_*^{(1)}$ for the angle of enclosure of the contour L by a plastic zone is found from (3.5) taking (3.9) into account

$$\sin\theta_*^{(1)}=\pm\frac{1}{2}\varepsilon\sqrt{s_0/s_1} \tag{3.10}$$

The boundary conditions to determine the third approximation of the solution of the problem under consideration are written according to (3.2) and (3.7) in the form

$$\sigma_r^{(3)}|_{L_*}=0,\quad \tau_{r\theta}^{(3)}|_{L_*}=-8\varkappa s_1\varepsilon^{-1}\sin\theta,\quad \sigma_{11}^{\infty}=\sigma_{22}^{\infty}=0, \tag{3.11}$$
$$\sigma_\theta^{(3)}|_{L_*}=2s_2r_3,\quad \varkappa=\operatorname{sign}(\cos\theta)$$

On the segment $[-\pi,\pi]$ the $r_{r\theta}^{(3)}|_{r=1}$ is an odd function. Its Fourier series expansion will have the form

$$\tau_{r\theta}^{(3)}|_{r=1}=\sum_{m=1}^{\infty}b_{2m}\cos 2m\theta \tag{3.12}$$
$$b_{2m}=\frac{2}{\pi}\left\{C\left[\frac{\sin(2m-1)\theta_*^{(1)}}{2m-1}-\frac{\sin(2m+1)\theta_*^{(1)}}{2m+1}\right]+\right.$$
$$\left.D\left[\frac{\sin(2m-3)\theta_*^{(1)}}{2m-3}-\frac{\sin(2m+3)\theta_*^{(1)}}{2m+3}\right]\right\}$$
$$C=4\frac{s_1}{s_2}\left(3\frac{s_1}{\varepsilon^3}-\frac{s_0}{\varepsilon}\right),\quad D=-4\frac{s_1^2}{s_2\varepsilon^3}$$

The solution of (1.1) with the boundary conditions (3.11) and (3.12) is sought by using (1.3) in which the functions $\Phi(z)$ and $\Psi(z)$ are determined by the relations /5/

$$\Phi(z)=-\frac{1}{2}\sum_{m=1}^{\infty}b_{2m}\frac{1}{z^{2m}},\quad \Psi(z)=-\sum_{m=1}^{\infty}b_{2m}(m+1)\frac{1}{z^{2m+2}}$$

and, therefore, the stress components have the form

$$\sigma_r^{(3)}=-\sum_{m=1}^{\infty}(m+1)b_{2m}\frac{1}{r^{2m}}\left(1-\frac{1}{r^2}\right)\cos 2m\theta \tag{3.13}$$
$$\sigma_\theta^{(3)}=\sum_{m=1}^{\infty}(m+1)b_{2m}\frac{1}{r^{2m}}\left(\frac{m-1}{m+1}-\frac{1}{r^2}\right)\cos 2m\theta$$
$$\tau_{r\theta}^{(3)}=-\sum_{m=1}^{\infty}(m+1)b_{2m}\frac{1}{r^{2m}}\left(\frac{m}{m+1}-\frac{1}{r^2}\right)\sin 2m\theta$$

Taking account of (3.5), (3.12) and (3.13), we determine the unknown function r_3 from the fifth boundary condition in (3.11)

$$r_3=\frac{8}{\pi}\frac{s_1^2}{s_2^2}\left[\frac{s_0}{s_1}\frac{\sin\theta_*^{(1)}}{\varepsilon}-\frac{4}{3}\frac{\sin^3\theta_*^{(1)}}{\varepsilon^3}-4\frac{\sin\theta_*^{(1)}\sin^2\theta}{\varepsilon^3}\right]+O(\varepsilon)$$

The second approximation $\theta_*^{(2)}$ for the angle of enclosure of the contour L by a plastic domain is determined from the relationship

$$\sin\theta_*^{(2)}=\pm\left(\frac{1}{2}\varepsilon\sqrt{\frac{s_0}{s_1}}-\varepsilon^2\frac{2s_0}{3\pi s_2}\right)$$

In the case of uniaxial tension $(P_{10}=0)$ of an elasto-plastic plane with a circular hole, we have

$$r_2=\frac{1}{10}\left(3-4\frac{\sin^2\theta}{\varepsilon^2}\right),\quad \sin\theta_*^{(1)}=\pm\frac{\sqrt{3}}{2}\varepsilon$$
$$r_3=\frac{8\sqrt{3}}{25\pi}\left(1-2\frac{\sin^2\theta}{\varepsilon^2}\right),\quad \sin\theta_*^{(3)}=\pm\left(\varepsilon\frac{\sqrt{3}}{2}-\varepsilon^2\frac{2}{5\pi}\right)$$

The position of the elasto-plastic boundaries L_s is shown in Fig.2 for the case of uniaxial tension of a plane with a circular hole, while graphs are given in Fig.3 for the dependence $[\sigma_r]/2k$ (the solid lines), $[\sigma_\theta]/2k$ (the dashes), $[\tau_{r\theta}]/2k$ (the dash-dot lines) on $T=\sin\theta/\sin\theta_*$ along the boundary L_s for values of the small loading parameter $\varepsilon=0.3$; 0.4; 0.6 (curves *1-3*, respectively).

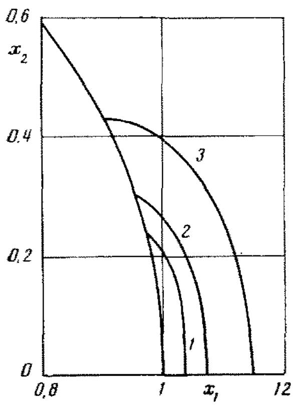

Fig.2

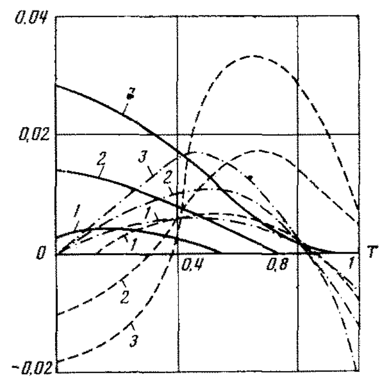

Fig.3

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